Annals of Combinatorics manuscript No.
(will be inserted by the editor)

A valley version of the Delta square conjecture

Alessandro Iraci · Anna Vanden Wyngaerd

Received: date / Accepted: date

Abstract Inspired by [Qiu, Wilson 2019] and [D'Adderio, Iraci, Vanden Wyngaerd 2019 - Delta Square], we formulate a generalised Delta square conjecture (valley version). Furthermore, we use similar techniques as in [Haglund, Sergel 2019] to obtain a schedule formula for the combinatorics of our conjecture. We then use this formula to prove that the (generalised) valley version of the Delta conjecture implies our (generalised) valley version of the Delta square conjecture. This implication broadens the argument in [Sergel 2016], relying on the formulation of the touching version in terms of the Θ_f operators introduced in [D'Adderio, Iraci, Vanden Wyngaerd 2019 - Theta Operators].

Keywords Dyck paths · Delta conjecture

Mathematics Subject Classification (2010) 05E05 · 05A19

1 Introduction

In [13], Haglund, Remmel and Wilson conjectured a combinatorial formula for $\Delta'_{e_{n-k-1}} e_n$ in terms of decorated labelled Dyck paths, which they called *Delta conjecture* after the so called Delta operators Δ'_f introduced by Bergeron, Garsia, Haiman, and Tesler [2] for any symmetric function f. There are two versions of the conjecture, referred to as the *rise* and the *valley* version.

In the same article [13] the authors conjecture a combinatorial formula for the more general expression $\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n$ in terms of decorated partially labelled Dyck paths, which we call *generalised Delta conjecture* (rise version). In this paper, the authors also state a *touching* refinement (where the number of times the Dyck path returns to the main diagonal is specified) of their conjecture. In [9], the authors introduce the Θ_f operators, and reformulate the touching version using these tools. In the present work, we will be using the latter formulation.

The Delta conjecture and its derivatives have attracted considerable attention since their formulation, see among others [5–8, 10, 14, 15, 20–22, 24–26]. Most of the earlier work concerns the rise version, but interest in the valley version is growing.

The special case $k = 0$ of the Delta conjecture, which is known as the *shuffle conjecture* [12], was recently proved by Carlsson and Mellit [4]. The shuffle theorem, thanks to the famous $n!$ *conjecture*, now $n!$ theorem of Haiman [16], gives a combinatorial formula for the Frobenius characteristic of the $\mathfrak{S}_n$-module of diagonal harmonics studied by Garsia and Haiman.

In [18] Loehr and Warrington conjecture a combinatorial formula for $\Delta_{e_n} \omega(p_n) = \nabla \omega(p_n)$ in terms of labelled square paths (ending east), called the *square conjecture*. The special case $\langle \cdot, e_n \rangle$ of this conjecture,

A. Iraci
Université Libre de Bruxelles (ULB)
Département de Mathématique
Boulevard du Triomphe, B-1050 Bruxelles
Belgium E-mail: airaci@ulb.ac.be

A. Vanden Wyngaerd
Université Libre de Bruxelles (ULB)
Département de Mathématique
Boulevard du Triomphe, B-1050 Bruxelles
Belgium E-mail: anvdwyng@ulb.ac.be

known as the q,t*-square*, was proved by Can and Loehr in [3]. Recently Sergel [23] proved the full square conjecture, by showing that the shuffle theorem by Carlsson and Mellit [4] implies the square conjecture (now square theorem).

In [6] the authors conjecture a combinatorial formula for $\frac{[n-k]_t}{[n]_t}\Delta_{h_m}\Delta_{e_{n-k}}\omega(p_n)$ in terms of *rise-decorated partially labelled square paths* that we call *generalised Delta square conjecture* (rise version). This conjecture extends the square conjecture of Loehr and Warrington [18] (now a theorem [23]), i.e. it reduces to that one for $m=k=0$. Moreover, it extends the generalised Delta conjecture in the sense that on decorated partially labelled Dyck paths the combinatorial statistics coincide.

In [20], the authors state a *generalised Delta conjecture* (valley version), extending the valley version of the Delta conjecture. They also prove the case $q=0$, extending the results in [7].

Inspired by [20] and [6], we formulate two statements that can reasonably be called the *generalised Delta square conjecture* (valley version). One is a combinatorial interpretation of the symmetric function

$$\frac{[n-k]_q}{[n]_q}\Delta_{h_m}\Delta_{e_{n-k}}\omega(p_n) = \frac{[n]_t}{[n-k]_t}\Delta_{h_m}\Theta_{e_k}\nabla\omega(p_{n-k})$$

(notice the swapping of q and t with respect to the rise version). The other is an interpretation of $\Delta_{h_m}\Theta_{e_k}\nabla\omega(p_{n-k})$, for which the combinatorics seems to be more natural and does not have the multiplicative factor.

Next, we adapt the schedule formula in [15] to objects with repeated labels, which enabled us to incorporate the monomials into the formula. This allowed us to obtain a schedule formula for the combinatorics of our conjecture and to deal with the symmetric functions more easily. As a byproduct, our formula provides a new factorisation of all other previous schedule formulae concerning Dyck or square paths.

Finally, we use this formula to prove that the (generalised) valley version of the Delta conjecture implies our (generalised) valley version of the Delta square conjecture. This implication broadens the argument in [23], relying on the formulation of the touching version in terms of the Θ_f operators.

In [9], the authors proved the case $k=0$ of the generalised Delta conjecture and so the implication establishes the case $k=0$ of the generalised square conjecture.

2 Combinatorial definitions

Definition 1 A *square path* of size n is a lattice path going from $(0,0)$ to (n,n) consisting of east or north unit steps, always ending with an east step. The set of such paths is denoted by $\mathsf{SQ}(n)$. The *shift* of a square path is the maximum value s such that the path intersect the line $y=x-s$ in at least one point. We refer to the line $y=x+i$ as i*-th diagonal* of the path and to the line $x=y$, (the 0-th diagonal) as the *main diagonal*. A vertical step whose starting point lies on the i-th diagonal is said to be at *height* i. A *Dyck path* is a square path whose shift is 0. The set of Dyck paths is denoted by $\mathsf{D}(n)$. Of course $\mathsf{D}(n) \subseteq \mathsf{SQ}(n)$.

For example, the path in Figure 1 has shift 3.

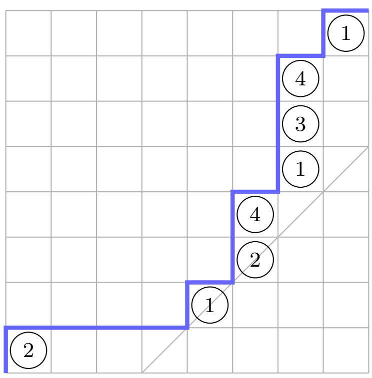


Fig. 1: Example of an element in $\mathsf{LSQ}(8)$.

Definition 2 Let π be a square path of size n. We define its *area word* to be the sequence of integers $a(\pi) = (a_1(\pi), a_2(\pi), \cdots, a_n(\pi))$ such that the i-th vertical step of the path starts from the diagonal $y = x + a_i(\pi)$. For example the path in Figure 1 has area word (0, –3, –3, –2, –2, –1, 0, 0).

Definition 3 A *partial labelling* of a square path π of size n is an element $w \in \mathbb{N}^n$ such that

- if $a_i(\pi) > a_{i-1}(\pi)$, then $w_i > w_{i-1}$,
- $a_1(\pi) = 0 \implies w_1 > 0$,
- there exists an index i such that $a_i(\pi) = -\mathsf{shift}(\pi)$ and $w_i(\pi) > 0$,

i.e. if we label the i-th vertical step of π with w_i, then the labels appearing in each column of π are strictly increasing from bottom to top, with the additional restrictions that, if the path starts north then the first label cannot be a 0, and that there is at least one positive label lying on the base diagonal.

We omit the word *partial* if the labelling is composed of strictly positive labels only.

Definition 4 A *(partially) labelled square path* (resp. *Dyck path*) is a pair (π, w) where π is a square path (resp. Dyck path) and w is a (partial) labelling of π. We denote by $\mathsf{LSQ}(m, n)$ (resp. $\mathsf{LD}(m, n)$) the set of labelled square paths (resp. Dyck paths) of size $m + n$ with exactly n positive labels, and thus exactly m labels equal to 0.

The following definitions will be useful later on.

Definition 5 Let w be a labelling of square path of size n. We define $x^w \coloneqq \prod_{i=1}^n x_{w_i}|_{x_0=1}$.

The fact that we set $x_0 = 1$ explains the use of the expression *partially labelled*, as the labels equal to 0 do not contribute to the monomial.

Sometimes we will, with an abuse of notation, write π as a shorthand for a labelled path (π, w). In that case, we use the identification $x^\pi \coloneqq x^w$.

Now we want to extend our sets introducing some decorations.

Definition 6 The *contractible valleys* of a labelled square path π are the indices $1 \leq i \leq n$ such that one of the following holds:

- $i = 1$ and either $a_1(\pi) < -1$, or $a_1(\pi) = -1$ and $w_1 > 0$,
- $i > 1$ and $a_i(\pi) < a_{i-1}(\pi)$,
- $i > 1$ and $a_i(\pi) = a_{i-1}(\pi) \wedge w_i > w_{i-1}$.

We define

$$v(\pi, w) \coloneqq \{1 \leq i \leq n \mid i \text{ is a contractible valley}\},$$

corresponding to the set of vertical steps that are directly preceded by a horizontal step and, if we were to remove that horizontal step and move it after the vertical step, we would still get a square path with a valid labelling. In particular, if the vertical step is in the first row and it is attached to a 0 label, then we require that it is preceded by at least two horizontal steps (as otherwise by removing it we get a path starting north with a 0 label in the first row).

Remark 1 These slightly contrived conditions on the steps labelled 0 have a more natural formulation in terms of steps labelled ∞, see Section 7.

This extends the definition of contractible valley given in [13] to (partially) labelled square paths.

Definition 7 The *rises* of a (labelled) square path π are the indices

$$r(\pi) \coloneqq \{2 \leq i \leq n \mid a_i(\pi) > a_{i-1}(\pi)\},$$

i.e. the vertical steps that are directly preceded by another vertical step.

Definition 8 A *valley-decorated (partially) labelled square path* is a triple (π, w, dv) where (π, w) is a (partially) labelled square path and $dv \subseteq v(\pi, w)$. A *rise-decorated (partially) labelled square path* is a triple (π, w, dr) where (π, w) is a (partially) labelled square path and $dr \subseteq r(\pi)$.

Again, we will often write π as a shorthand for the corresponding triple (π, w, dv) or (π, w, dr).

We denote by $\mathsf{LSQ}(m,n)^{\bullet k}$ (resp. $\mathsf{LSQ}(m,n)^{*k}$) the set of partially labelled valley-decorated (resp. rise-decorated) square paths of size $m+n$ with n positive labels and k decorated contractible valleys (resp. decorated rises). We denote by $\mathsf{LD}(m,n)^{\bullet k}$ (resp. $\mathsf{LD}(m,n)^{*k}$) the corresponding subsets of Dyck paths.

We also define $\mathsf{LSQ}'(m,n)^{\bullet k}$ as the set of paths in $\mathsf{LSQ}(m,n)^{\bullet k}$ such that there exists an index i such that $a_i(\pi) = -\mathsf{shift}(\pi)$ and $i \notin dv \wedge w_i(\pi) > 0$, i.e. there is at least one positive label lying on the bottom-most diagonal that is not a decorated valley. The importance of this set will be evident later in the paper.

Finally, we sometimes omit writing m or k when they are equal to 0. Notice that, because of the restrictions we have on the labelling and the decorations, the only path with $n = 0$ is the empty path, for which also $m = 0$ and $k = 0$.

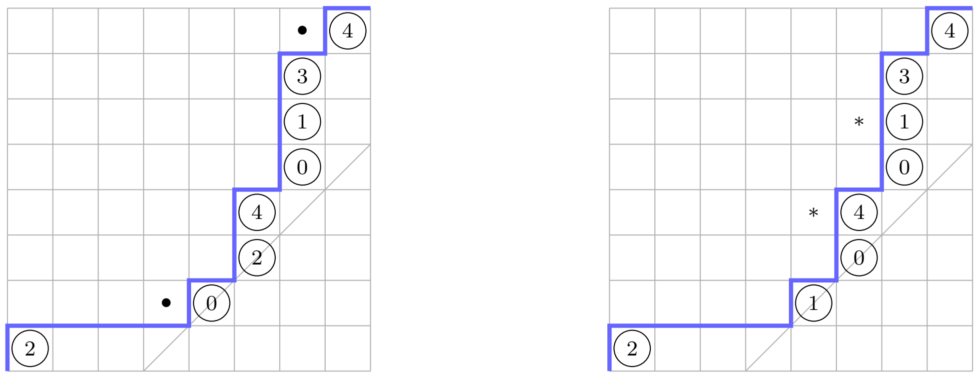


Fig. 2: Example of an element in $\mathsf{LSQ}(2,6)^{\bullet 2}$ (left) and an element in $\mathsf{LSQ}(2,6)^{*2}$ (right).

We define two statistics on this set that reduce to the ones defined in [18] when $m = 0$ and $k = 0$.

Definition 9 Let $(\pi, w, dr) \in \mathsf{LSQ}(m,n)^{*k}$ and s be its shift. We define

$$\mathsf{area}(\pi, w, dr) \coloneqq \sum_{i \notin dr} (a_i(\pi) + s),$$

i.e. the number of whole squares between the path and the base diagonal that are not in rows containing a decorated rise.

For $(\pi, w, dv) \in \mathsf{LSQ}(m,n)^{\bullet k}$, we define $\mathsf{area}(\pi, w, dv) \coloneqq \mathsf{area}(\pi, w, \varnothing)$, where $(\pi, w, \varnothing) \in \mathsf{LSQ}(m,n)^{*0}$.

For example, the paths in Figure 2 have area 13 (left) and 10 (right). Notice that the area does not depend on the labelling.

Definition 10 Let $(\pi, w, dv) \in \mathsf{LSQ}(m,n)^{\bullet k}$. For $1 \leq i < j \leq n$, the pair (i,j) is a *diagonal inversion* if

- either $a_i(\pi) = a_j(\pi)$ and $w_i < w_j$ (*primary inversion*),
- or $a_i(\pi) = a_j(\pi) + 1$ and $w_i > w_j$ (*secondary inversion*),

where w_i denotes the i-th letter of w, i.e. the label of the vertical step in the i-th row. Then we define

$$\mathsf{dinv}(\pi) \coloneqq \#\{1 \leq i < j \leq n \mid (i,j) \text{ inversion } \wedge i \notin dv\} + \#\{1 \leq i \leq n \mid a_i(\pi) < 0 \wedge w_i > 0\} - \#dv$$

where again π is a shorthand for (π, w, dv).

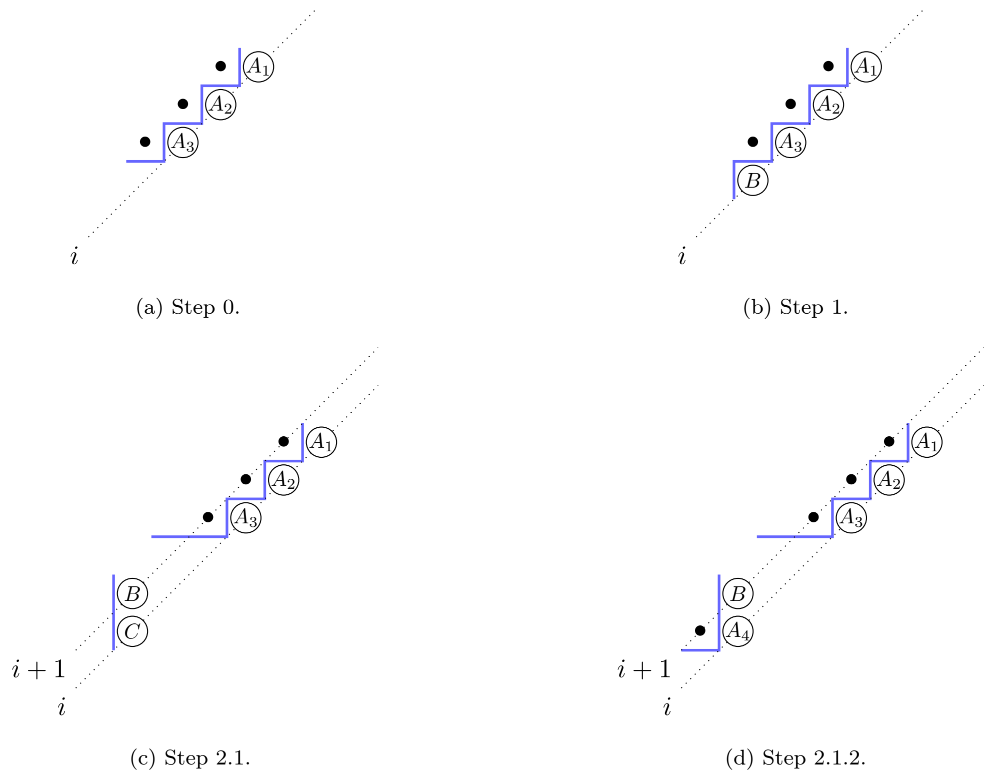
For $(\pi, w, dr) \in \mathsf{LSQ}(m,n)^{*k}$, we define $\mathsf{dinv}(\pi, w, dr) \coloneqq \mathsf{dinv}(\pi, w, \varnothing)$ where $\pi, w, \varnothing) \in \mathsf{LSQ}(m,n)^{\bullet 0}$.

We refer to the middle term, counting the non-zero labels below the main diagonal, as *bonus* or *tertiary dinv*.

For example, the path in Figure 2 (left) has dinv equal to 4: 2 primary inversions in which the leftmost label is not a decorated valley, i.e. $(1,7)$ and $(1,8)$; 1 secondary inversion in which the leftmost label is not a decorated valley, i.e. $(1,6)$; 3 bonus dinv, coming from the rows 3, 4, and 6; 2 decorated valleys.

It is not immediately obvious why the dinv of a valley-decorated path is always non-negative.

Proposition 1 *For all* $\pi \in \mathsf{LSQ}(m,n)^{\bullet k}$, $\mathsf{dinv}(\pi) \geq 0$.



(a) Step 0.

(b) Step 1.

(c) Step 2.1.

(d) Step 2.1.2.

Fig. 3: Dinv is non-negative.

Proof We will show that each decorated valley of π implies at least one unit of primary, secondary or bonus dinv, that compensates the negative contribution of the valley itself.

Consider a decorated valley at height i. By definition, it is preceded by a horizontal step. For the remainder of the proof, "decorated valley" will refer to the decorated vertical step *and* the horizontal step that precedes it.

Step 0. Suppose the valley is part of a string of decorated valleys, labelled $A_s, \dots, A_1$ from left to right, see Figure 3a. Since the valleys are contractible we must have $A_s < \cdots < A_1$. This string is then directly preceded either by a vertical step that is not a decorated valley (as otherwise this would be part of the string), or by a horizontal step.

Step 1. If the string is preceded by a vertical step, then this step's label, say B, must be such that $B < A_s < \cdots < A_1$ since the step labelled by A_s is a contractible valley, see Figure 3b. Thus, the step labelled B creates primary dinv with each of the decorated valleys in the string following it.

Step 2. If the string is preceded by a horizontal step, consider two subcases.

Step 2.1. First suppose that the valley labelled A_s is preceded by a leftmost vertical step at height i that is not a decorated valley (which is always true for $i \geq 0$). This implies that the step labelled A_s must be preceded at some point by two consecutive vertical steps, at height i and $i+1$, labelled C and B respectively, see Figure 3c. For all j, if $B > A_j$, then the step labelled B creates secondary dinv with the step labelled A_j. If $B \leq A_j$ then $C < A_j$.

Step 2.1.1. If the step labelled C is not a decorated valley then it creates primary dinv with the step labelled A_j, for all j.

Step 2.1.2. If, however the step labelled C is a decorated valley, rename its label A_{s+1} and consider it as part of the "string" of decorated valleys, see Figure 3d. Restart the argument from Step 1 (since the path is finite, this loop must terminate).

Step 2.2. The step labelled A_s is not preceded by a vertical step at height i that is not a decorated valley. This implies that $i < 0$. Thus, decorated valleys at height i that are not labelled 0 contribute to the bonus dinv. So we are exclusively concerned with the decorated valleys labelled 0. Decorated valleys labelled 0 that are not the first step at height i must create secondary dinv with a step to its left: indeed, they must be preceded by two horizontal steps, otherwise they would not be contractible. Since they are not the first step at height i, they must be preceded by two consecutive vertical steps, at height i and $i+1$, labelled B and C respectively, as in Figure 3c. Since B labels a rise, it must be positive and therefore must create secondary dinv with steps labelled 0 to its right.

Thus, we are left with a decorated valley labelled 0 that is the first step at height i. By the definition of a contractible valley, this implies that $i \neq 1$. Since the square path must end east, there must be a rise at height $i+1<0$, which cannot be a decorated valley (as it is a rise), and that creates one unit of bonus dinv that is not compensating for any other decorated valley.

Finally, we recall two classical definitions.

Definition 11 Let $p_1, \dots, p_k$ be a sequence of integers. We define its *descent set*

$$\mathsf{Des}(p_1, \dots, p_k) \coloneqq \{1 \leq i \leq k-1 \mid p_i > p_{i+1}\}$$

and its *major index* $\mathsf{maj}(p_1, \dots, p_k)$ to be the sum of the elements of the descent set.

3 Symmetric functions

For all the undefined notations and the unproven identities, we refer to [5, Section 1], where definitions, proofs, and/or references can be found.

We denote by Λ the graded algebra of symmetric functions with coefficients in $\mathbb{Q}(q,t)$, and by $\langle\, , \rangle$ the *Hall scalar product* on Λ, defined by declaring that the Schur functions form an orthonormal basis.

The standard bases of the symmetric functions that will appear in our calculations are the monomial $\{m_\lambda\}_\lambda$, complete homogeneous $\{h_\lambda\}_\lambda$, elementary $\{e_\lambda\}_\lambda$, power $\{p_\lambda\}_\lambda$ and Schur $\{s_\lambda\}_\lambda$ bases.

For a partition $\mu \vdash n$, we denote by

$$\widetilde{H}_\mu \coloneqq \widetilde{H}_\mu[X] = \widetilde{H}_\mu[X;q,t] = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda\mu}(q,t) s_\lambda$$

the *(modified) Macdonald polynomials*, where

$$\widetilde{K}_{\lambda\mu} \coloneqq \widetilde{K}_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)}$$

are the *(modified) Kostka coefficients* (see [11, Chapter 2] for more details).

Macdonald polynomials form a basis of the ring of symmetric functions Λ. This is a modification of the basis introduced by Macdonald [19].

If we identify the partition μ with its Ferrers diagram, i.e. with the collection of cells $\{(i,j) \mid 1 \leq i \leq \mu_i, 1 \leq j \leq \ell(\mu)\}$, then for each cell $c \in \mu$ we refer to the *arm*, *leg*, *co-arm* and *co-leg* (denoted respectively as $a_\mu(c), l_\mu(c), a_\mu(c)', l_\mu(c)'$) as the number of cells in μ that are strictly to the right, above, to the left and below c in μ, respectively.

Let $M \coloneqq (1-q)(1-t)$. For every partition μ, we define the following constants:

$$\begin{aligned}
B_\mu &\coloneqq B_\mu(q,t) = \sum_{c \in \mu} q^{a'_\mu(c)} t^{l'_\mu(c)}, \\
D_\mu &\coloneqq D_\mu(q,t) = MB_\mu(q,t) - 1, \\
T_\mu &\coloneqq T_\mu(q,t) = \prod_{c \in \mu} q^{a'_\mu(c)} t^{l'_\mu(c)} = q^{n(\mu')} t^{n(\mu)} = e_{|\mu|}[B_\mu], \\
\Pi_\mu &\coloneqq \Pi_\mu(q,t) = \prod_{c \in \mu/(1,1)} (1 - q^{a'_\mu(c)} t^{l'_\mu(c)}), \\
w_\mu &\coloneqq w_\mu(q,t) = \prod_{c \in \mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}).
\end{aligned}$$

We will make extensive use of the *plethystic notation* (cf. [11, Chapter 1]).

We need to introduce several linear operators on Λ.

Definition 12 ([1, 3.11]) We define the linear operator $\nabla \colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as

$$\nabla \widetilde{H}_\mu = T_\mu \widetilde{H}_\mu.$$

Definition 13 We define the linear operator $\mathbf{\Pi}\colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as

$$\mathbf{\Pi}\widetilde{H}_\mu = \Pi_\mu \widetilde{H}_\mu$$

where we conventionally set $\Pi_\varnothing := 1$.

Definition 14 For $f \in \Lambda$, we define the linear operators $\Delta_f, \Delta'_f\colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as

$$\Delta_f \widetilde{H}_\mu = f[B_\mu]\widetilde{H}_\mu, \qquad \Delta'_f \widetilde{H}_\mu = f[B_\mu - 1]\widetilde{H}_\mu.$$

Observe that on the vector space of symmetric functions homogeneous of degree n, denoted by $\Lambda^{(n)}$, the operator ∇ equals Δ_{e_n}.

We also introduce the Theta operators, first defined in [9]

Definition 15 For any symmetric function $f \in \Lambda^{(n)}$ we introduce the following *Theta operators* on Λ: for every $F \in \Lambda^{(m)}$ we set

$$\Theta_f F := \begin{cases} 0 & \text{if } n \geq 1 \text{ and } m = 0 \\ f \cdot F & \text{if } n = 0 \text{ and } m = 0 \\ \mathbf{\Pi} f^* \mathbf{\Pi}^{-1} F & \text{otherwise} \end{cases},$$

and we extend by linearly the definition to any $f, F \in \Lambda$.

It is clear that Θ_f is linear, and moreover, if f is homogenous of degree k, then so is Θ_f, i.e.

$$\Theta_f \Lambda^{(n)} \subseteq \Lambda^{(n+k)} \qquad \text{for } f \in \Lambda^{(k)}.$$

It is convenient to introduce the so called q-notation. In general, a q-analogue of an expression is a generalisation involving a parameter q that reduces to the original one for $q \to 1$.

Definition 16 For a natural number $n \in \mathbb{N}$, we define its q-analogue as

$$[n]_q := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Given this definition, one can define the q-factorial and the q-binomial as follows.

Definition 17 We define

$$[n]_q! := \prod_{k=1}^{n} [k]_q \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Definition 18 For x any variable and $n \in \mathbb{N} \cup \{\infty\}$, we define the *$q$-Pochhammer symbol* as

$$(x;q)_n := \prod_{k=0}^{n-1} (1 - xq^k) = (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}).$$

We can now introduce yet another family of symmetric functions.

Definition 19 For $0 \leq k \leq n$, we define the symmetric function $E_{n,k}$ by the expansion

$$e_n\left[X \frac{1-z}{1-q}\right] = \sum_{k=0}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}.$$

Notice that $E_{n,0} = \delta_{n,0}$. Setting $z = q^j$ we get

$$e_n\left[X \frac{1-q^j}{1-q}\right] = \sum_{k=0}^{n} \frac{(q^j;q)_k}{(q;q)_k} E_{n,k} = \sum_{k=0}^{n} \begin{bmatrix} k+j-1 \\ k \end{bmatrix}_q E_{n,k}$$

and in particular, for $j = 1$, we get

$$e_n = E_{n,0} + E_{n,1} + E_{n,2} + \cdots + E_{n,n},$$

so these symmetric functions split e_n, in some sense.

We care in particular about the following identity.

Proposition 2 ([3, Theorem 4]) *For $n > 0$,*

$$\omega(p_n) = \sum_{k=1}^{n} \frac{[n]_q}{[k]_q} E_{n,k}.$$

The Theta operators will be useful to restate the Delta conjectures in a new fashion, thanks to the following results.

Theorem 1 ([9, Theorem 3.1]) *For $n > 0$,*

$$\Theta_{e_k} \nabla e_{n-k} = \Delta'_{e_{n-k-1}} e_n.$$

Theorem 2 ([9, Theorem 3.3]) *For $n > 0$,*

$$\frac{[n]_q}{[n-k]_q} \Theta_{e_k} \nabla \omega(p_{n-k}) = \frac{[n-k]_t}{[n]_t} \Delta_{e_{n-k}} \omega(p_n).$$

Corollary 1 *For $n > 0$,*

$$\frac{[n]_t}{[n-k]_t} \Theta_{e_k} \nabla \omega(p_{n-k}) = \frac{[n-k]_q}{[n]_q} \Delta_{e_{n-k}} \omega(p_n).$$

4 Delta conjectures

By *Delta conjectures* we refer to a family of conjectures that provide a combinatorial interpretation of certain symmetric functions that arise from the Delta operators and show positivity properties.

The first and most famous of the Delta conjectures is known as *shuffle conjecture*, now a theorem by E. Carlsson and A. Mellit (see [4]).

Theorem 3 **(Shuffle theorem** [4, Theorem 7.5]**)**

$$\nabla e_n = \sum_{\pi \in \mathsf{LD}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi}.$$

The shuffle theorem is especially important because ∇e_n has another interpretation, as the bigraded Frobenius characteristic of the S_n module of the *diagonal harmonics*. This is one of the facts that first motivated the study of Macdonald polynomials, and it has been proved by M. Haiman in [17]. See also [16] for further details.

The *Delta conjecture* is a generalisation of the shuffle conjecture, introduced by J. Haglund, J. Remmel, and A. Wilson in [13]. In the same paper, the authors suggest that an even more general conjecture should hold, which we call *generalised Delta conjecture*. It reads as follows.

Conjecture 1 ((Generalised) Delta conjecture, valley version [20, Conjecture 1.3]*)*

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{\pi \in \mathsf{LD}(m,n)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi}.$$

For $m = 0$ this conjecture first appears together with the rise version in [13]. The full statement, together with a proof of the case $q = 0$, has been given by D. Qiu and A. Wilson in [20].

Conjecture 2 ((Generalised) Delta conjecture, rise version [13, Conjecture 7.4]*)*

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{\pi \in \mathsf{LD}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi}.$$

The rise version of the Delta conjecture is simply the case $m = 0$ of the general case.

Recalling that $\nabla|_{\Lambda^{(n)}} = \Delta'_{e_{n-1}}|_{\Lambda^{(n)}}$, it is clear that for $k = 0$ both the versions of the Delta conjecture reduce to the shuffle theorem.

The *square conjecture* was first suggested by N. Loehr and G. Warrington in [18], and it was then proved by E. Sergel in [23] using the shuffle theorem.

Theorem 4 **(Square Theorem** [23, Theorem 4.11]**)**

$$\nabla\omega(p_n) = \sum_{\pi\in\mathsf{LSQ}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

Unfortunately, adding zero labels and decorated rises to square paths in the trivial way and q, t-counting the resulting objects with respect to the bistatistic $(\mathsf{dinv}, \mathsf{area})$ gives a polynomial that does not match the expected symmetric function. This issue has been addressed by M. D'Adderio and the authors, who stated the *generalised Delta square conjecture* in [6].

Conjecture 3 ((Generalised) Delta square conjecture, rise version [6, Conjecture 3.12]*)*

$$\frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \sum_{\pi\in\mathsf{LSQ}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

where the rise version of the square conjecture is simply the case $m = 0$ of the general case.

The square conjectures used to lack a valley version. Computational evidence suggests the following, checked by computer up to $n = 6$.

Conjecture 4 ((Generalised) Delta square conjecture, valley version)

$$\frac{[n-k]_q}{[n]_q} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \sum_{\pi\in\mathsf{LSQ}(m,n)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

Notice that the symmetric function we propose for the valley version differs from the one appearing in the rise version (for $m = 0$) as the multiplicative factor is the ratio of two q-analogues instead of two t-analogues. This suggests a potential extension of the conjecture to a version with both decorated rises and contractible valleys, possibly using the Theta operators appearing in [9]. The power series associated to the obvious combinatorial extension, however, seems to be quasi-symmetric function which is not symmetric, and thus further investigation is required to find suitable statistics.

We can restate it in terms of Theta operators as follows.

Conjecture 5 ((Generalised) Delta square conjecture, valley version)

$$\frac{[n]_t}{[n-k]_t} \Delta_{h_m} \Theta_{e_k} \nabla\omega(p_{n-k}) = \sum_{\pi\in\mathsf{LSQ}(m,n)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

We need to state a refinement of the Delta conjecture, valley version, that naturally arises when stating it in terms of Theta operators. But first, we need another combinatorial definition. Let

$$\mathsf{LSQ}(m, n\backslash r)^{\bullet k} := \{\pi \in \mathsf{LSQ}(m,n)^{\bullet k} \mid \#\{i \notin dv \colon a_i = -\mathsf{shift}(\pi) \wedge w_i > 0\} = r\},$$

which is the set of labelled valley-decorated square paths of size $m + n$ with m labels equal to 0 and k decorations such that there are exactly r steps which are neither 0 labels nor decorated valleys on the bottom-most diagonal, and let

$$\mathsf{LD}(m, n\backslash r)^{\bullet k} := \mathsf{LSQ}(m, n\backslash r)^{\bullet k} \cap \mathsf{LD}(m,n)^{\bullet k},$$

which is the subset of corresponding labelled valley-decorated Dyck paths. We state the following.

Conjecture 6 (Touching Delta conjecture, valley version)

$$\Theta_{e_k} \nabla E_{n-k,r} = \sum_{\pi\in\mathsf{LD}(n\backslash r)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

It is immediate that Conjecture 6 implies the case $m = 0$ of Conjecture 1, as it is enough to sum over r and then apply Theorem 1.

We need to state the same refinement for the generalised version too.

Conjecture 7 (Generalised touching Delta conjecture, valley version)

$$\Delta_{h_m} \Theta_{e_k} \nabla E_{n-k,r} = \sum_{\pi\in\mathsf{LD}(m,n\backslash r)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

Remark 2 In [9, Theorem 7.1], the authors proved the case $k=0$ of Conjecture 7: the *generalised touching shuffle theorem.*

We now want to state yet another version of the Delta square conjecture, using the set $\mathsf{LSQ}'(m,n)^{\bullet k}$ previously introduced.

Conjecture 8 (Modified Delta square conjecture, valley version)

$$\Theta_{e_k}\nabla\omega(p_{n-k}) = \sum_{\pi\in\mathsf{LSQ}'(n)^{\bullet k}} q^{\mathsf{dinv}(\pi)}t^{\mathsf{area}(\pi)}x^{\pi}.$$

This conjecture is new and it is more nice-looking than the other forms of the Delta square conjecture as it does not have any multiplicative correcting factor. It also extends nicely to the $m>0$ case, as follows.

Conjecture 9 (Modified generalised Delta square conjecture, valley version)

$$\Delta_{h_m}\Theta_{e_k}\nabla\omega(p_{n-k}) = \sum_{\pi\in\mathsf{LSQ}'(m,n)^{\bullet k}} q^{\mathsf{dinv}(\pi)}t^{\mathsf{area}(\pi)}x^{\pi}.$$

Our goal is to show that Conjecture 7 implies Conjecture 9, and as a corollary that Conjecture 6 implies Conjecture 8.

5 Schedule numbers for repeated labels

Definition 20 Let (π, w, dv) be a valley-decorated labelled square path with shift s. For $i\geq 0$ we set ρ_i to be the word consisting of the labels appearing in the $(i-s)$-th diagonal, marked with a $\bullet$ if it labels a decorated valley, in increasing order, where we consider $c<\overset{\bullet}{c}<c+1$. The *diagonal word* of (π,w,dv) is $\mathsf{dw}(\pi,w,dv) \coloneqq \rho_\ell\dots\rho_1\rho_0$.

For such a word z we define $\mathsf{maj}(z)$ and x^z as usual, ignoring the decorations.

For example the diagonal word of the path in Figure 4 is $1243\overset{\bullet}{1}41\overset{\bullet}{1}$. Notice that the ρ_i are the runs of $\mathsf{dw}(\pi,w,dv)$, i.e. the maximal weakly increasing substrings (disregarding decorations).

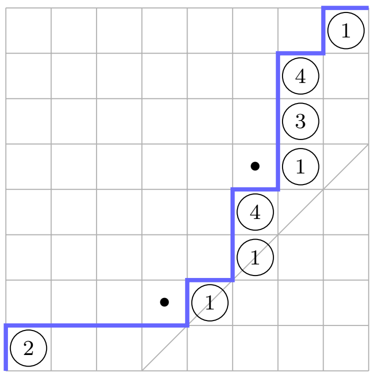


Fig. 4: Square path with diagonal word $1243\overset{\bullet}{1}41\overset{\bullet}{1}$

Remark 3 If w is the diagonal word of some path in $\mathsf{LSQ}(m,n)^{\bullet k}$ it is a decorated word with letters in the alphabet $\mathbb{N}$. For a decorated word w, we define $\mathsf{maj}(w)$ and x^w to be computed as usual, simply ignoring the decorations.

Definition 21 Let $z \coloneqq \mathsf{dw}(\pi,w,dv)$ be the diagonal word of a valley-decorated labelled square path (π,w,dv) such that $z=\rho_\ell\cdots\rho_0$, where the ρ_i's are its runs. We define its *i-th run multiplicity functions* $z_i, z_i^\bullet : \mathbb{N}\to\mathbb{N}$, where for any $c\in\mathbb{N}$

$$\begin{aligned} z_i(c) &= \#\text{ of undecorated } c\text{'s in } \rho_i \\ z_i^\bullet(c) &= \#\text{ of decorated } c\text{'s in } \rho_i. \end{aligned}$$

Clearly, each function z_i has finite support.

Definition 22 Consider $\pi \in \mathsf{LSQ}(m,n)^{\bullet k}$ and set $\mathsf{dw}(\pi) = \rho_\ell \cdots \rho_0$, where the ρ_i's are its runs. For $c \in \mathbb{N}$, we define its *schedule numbers* $w_{i,s}(c)$ as follows:

$$w_{i,s}(c) \coloneqq \begin{cases} \sum_{d>c} z_i(d) + \sum_{d<c} z_{i-1}(d) & \text{if } i \in \{s+1,\dots,\ell\} \\ \sum_{d>c} z_i(d) + 1 - \delta_{c,0} & \text{if } i = s \\ \sum_{d<c} z_i(d) + \sum_{d>c} z_{i+1}(d) & \text{if } i \in \{0,\dots,s-1\} \end{cases}$$
$$w^{\bullet}_{i,s}(c) \coloneqq \sum_{d<c} z_i(d) + \sum_{d>c} z_{i+1}(d) - \delta_{c,0}\delta_{i,s-1}.$$

Theorem 5 *Let $z = \rho_\ell \cdots \rho_0$ be the diagonal word of a path in $\mathsf{LSQ}'(m,n)^{\bullet k}$ so that the ρ_i are its runs. Let $b(z,s) \coloneqq \sum_{c>0} \left(\sum_{i<s} z_i(c)\right) + \sum_{i<s-1}(-z^{\bullet}_i(0))$. Then*

$$\sum_{\substack{\pi \in \mathsf{LSQ}(m,n)^{\bullet k} \\ \mathsf{shift}(\pi)=s \\ \mathsf{dw}(\pi)=z}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi = t^{\mathsf{maj}(z)} q^{b(z,s)} \prod_{i=0}^{\ell} \left(\prod_{c \geq 0} \begin{bmatrix} w_{i,s}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q q^{\binom{z^{\bullet}_i(c)}{2}} \begin{bmatrix} w^{\bullet}_{i,s}(c) \\ z^{\bullet}_i(c) \end{bmatrix}_q \right) x^z.$$

The proof of this result is similar to the one described by Haglund and Sergel in [15, Theorem 3.2] for Dyck paths, except that we consider repeated labels. For the sake of completeness, we repeat some of their arguments here.

Proof Let us begin by noting that since $z_i(c) = z^{\bullet}_i(c) = 0$ for all but a finite number of elements of $\mathbb{N}$ and thus all but a finite number of q-binomials of the right hand side are equal to 1, which means that the product is actually finite.

Next, observe that for any $\pi \in \mathsf{LSQ}(n)^{\bullet k}$ with $\mathsf{dw}(\pi) = z$ we trivially have $x^\pi = x^z$, so we only need to consider the q,t-enumerators. It is also not difficult to see that for any such path $\mathsf{maj}(z) = \mathsf{area}(\pi)$, indeed,

$$\begin{aligned} \mathsf{area}(\pi) &= \ell \cdot \#\rho_\ell + (\ell-1) \cdot \#\rho_{\ell-1} + \cdots + 1 \cdot \#\rho_1 \\ &= \rho_\ell + (\rho_\ell + \rho_{\ell-1}) + \cdots + (\rho_\ell + \rho_{\ell-1} + \cdots + \rho_1) = \mathsf{maj}(z). \end{aligned}$$

This takes care of the factor $t^{\mathsf{maj}(z)}$.

For the dinv, we will construct all the paths of a given diagonal word and shift, starting from the empty path, all the while keeping track of the dinv. We outline the different steps of the construction. We only describe the placement of the (decorated) labels in the lattice, as each such placement is the labelling of a unique square path.

1. For $i = s, s+1, \dots \ell$ insert the $z_i(c)$ labels equal to c into the $(i-s)$-th diagonal, for all $c \in \mathbb{N}$, in decreasing order.
2. For $i = s-1, s-2, \dots 0$ insert the $z_i(c)$ labels equal to c into the $(i-s)$-th diagonal, for all $c \in \mathbb{N}$, in increasing order.
3. For all i insert the $z^{\bullet}_i(c)$ decorated labels equal to c into the $(i-s)$-th diagonal, for all $c \in \mathbb{N}$, in decreasing order (the order of i is unimportant).

In other words in the first step we construct non-decorated Dyck paths, in the second we turn them into non-decorated square paths and in the third we add decorated labelled steps.

Call a (i,c)-*insertion* (respectively $(i,c)^{\bullet}$-*insertion*) the insertion of $z_i(c)$ (respectively $z^{\bullet}_i(c)$) labels equal to c into the i-th diagonal. We will now study for each insertion the numbers of ways it may be executed, and the contribution to the dinv each of these ways engenders.

We made figures illustrating the construction of some of the paths with diagonal word $44\ 223\ \overset{\bullet}{0}11\overset{\bullet}{2}$ and shift 1. We included them in Appendix A.

Dyck paths. First consider $i = s$. Right before the (s,c)-insertion, there are $\sum_{d>c} z_s(d)$ labels in the 0-th diagonal. If $c \neq 0$ the $z_i(c)$ labels may be inserted anywhere between these $\sum_{d>c} z_s(d) = w_{s,s}(c) - 1$ labels. If $c = 0$, since the leftmost label in the 0-th diagonal may never be 0, the c's may be inserted anywhere between the remaining $\sum_{d>c} z_s(d) - 1 = w_{s,s}(0) - 1$ labels. In both cases, any time one of the inserted c's precedes one of the d's with $d > c$, a unit of primary dinv is created. Thus the dinv of all possible insertions is q-counted by $\left[\begin{smallmatrix} w_{s,s}(c)+z_i(c)-1 \\ z_i(c) \end{smallmatrix}\right]_q$. See Figure 8 in Appendix A .

For $i > s$, consider the path right before the (i,c)-insertion. We identify two kinds of *insertion spots*:

1. a smaller label in the $(i-s-1)$-th diagonal, of which there are $\sum_{d<c} z_{i-1}(d)$;
2. a label in the $(i-s)$-th diagonal (which must be bigger than c because of the insertion order), of which there are $\sum_{d>c} z_i(d)$.

Thus, the total number of insertion spots comes to $w_{i,s}(c)$. Any (i,c)-insertion corresponds uniquely to an interlacing of the $w_{i,s}(c)-1$ insertion spots and $z_i(c)$ c's: indeed

- the first occurrence of c in the i-th diagonal must be preceded by an insertion spot;
- there is a unique way of inserting a string of consecutive c's right after any insertion spot. Say we want to insert k consecutive c's. Shift the labels following the insertion spot k squares to the north-east. Then insert the first of the string of c's into the square on top of an insertion spot of the first kind or in the square north-east of an insertion spot of the second kind;
- between two strings of consecutive c's there must be an insertion spot.

Any time an occurrence of c precedes an insertion spot of the first (respectively second) kind, a unit of secondary (respectively primary) dinv is created. So the dinv of all possible insertions is q-counted by $\begin{bmatrix} w_s(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q$. See Figure 9 in Appendix A .

Square paths. For the (i,c)-insertion with $i<s$ the insertions spots are

1. bigger labels in the $(i-s+1)$-th diagonal, of which there are $\sum_{d>c} z_{i+1}(d)$;
2. labels in the $(i-s)$-th diagonal (which are smaller than c due to the insertion order), of which there are $\sum_{d<c} z_i(d)$.

Thus there are $w_{i,s}(c)$ insertion spots. We have that

- the last occurrence of c must be followed by an insertion spot;
- there is an unique way of inserting a string of consecutive c's right *before* any insertion spot. Say we want to insert k consecutive c's. Shift the insertion spot and the labels following it k squares to the north-east. Insert the last of the string of c's in the square below an insertion spot of the first kind or in the square south-west of the insertion spot of the second kind;
- between two strings of consecutive c's there must be an insertion spot.

So the (i,c)-insertion corresponds uniquely to an interlacing of the $z_i(c)$ c's and the $w_{i,s}(c)-1$ insertion spots. An insertion spot of the first (respectively second) creates secondary (respectively primary) dinv with all following c's. Furthermore, any non-zero label that gets inserted under the main diagonal creates a unit of bonus dinv. Thus the dinv of all possible insertions is q-counted by $q^{(1-\delta_{c,0})z_i(c)}\begin{bmatrix} w_s(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q$. See Figure 10.

Decorations. Now we treat $(i,c)^\bullet$-insertions. Define the *dinv markers* of such an insertion to be the $\sum_{d<c} z_i(d)$ non-decorated labels smaller that c in the $(i-s)$-th diagonal and the $\sum_{d>c} z_{i+1}(d)$ non-decorated labels bigger than c in the the $(i-s+1)$-th diagonal. These dinv markers are exactly the labels with which a decorated c inserted to its right would create primary or secondary dinv.

First consider $i \geq s$. The number of dinv markers equals $w^\bullet_{i,s}(c)$. We claim that $(i,c)^\bullet$-insertions correspond bijectively to an interlacing of the $z^\bullet_i(c)$ inserted c's and $w^\bullet_{i,s}(c)$ dinv markers, starting with a dinv marker and without two consecutive c's. We show that this correspondence is a well defined and bijective.

Well defined. We have to show that for any $(i,c)^\bullet$-insertion, the corresponding interlacing has no two consecutive c's and starts with a dinv marker. In the proof of Proposition 1, it is argued that a decorated valley at height ≥ 0 is alway preceded by a label with which it creates primary or secondary dinv, i.e. a dinv marker. Next, we need to show that there may never be two consecutive c's in the interlacing, i.e. that there is always a dinv marker between two inserted c's. If the step labelled by an inserted c is followed by a vertical step, its label must be bigger that c and so it is a dinv marker. If it is followed by a horizontal step, it might be followed by a string of decorated labels at the same height: $B_1, \dots, B_l$. We must have $c < B_1 < \cdots < B_l$ since the valleys are contractible. If the step labelled B_l is followed by a vertical step, its label must be bigger than B_l and so a dinv marker. If the step labelled B_l is followed by a horizontal step the step after this horizontal step cannot be a decorated valley labelled c (not contractible) so it must either be a vertical, non-decorated step, or another horizontal step. In the latter case, the next label at height $i-s$ is a rise and so it is not decorated. Thus, there is a non-decorated label at height $i-s$ between our inserted decorated c and the next one. Again, we may use the arguments in the proof of Proposition 1 to conclude that there must be an dinv marker before the next occurrence of an inserted decorated c.

Injectivity. Suppose that there are two different insertions with the same interlacing of dinv markers and decorated inserted c's. This implies that between two (or after all) dinv markers there are two different ways to insert a decorated c. Combining these two ways, one would obtain a path with two inserted c's that are not separated by a dinv marker, in contradiction to what is shown in the previous paragraph.

Surjectivity. We must show that it is always possible to insert a decorated c between two (or after all) dinv markers. We describe an insertion procedure for all possibilities.

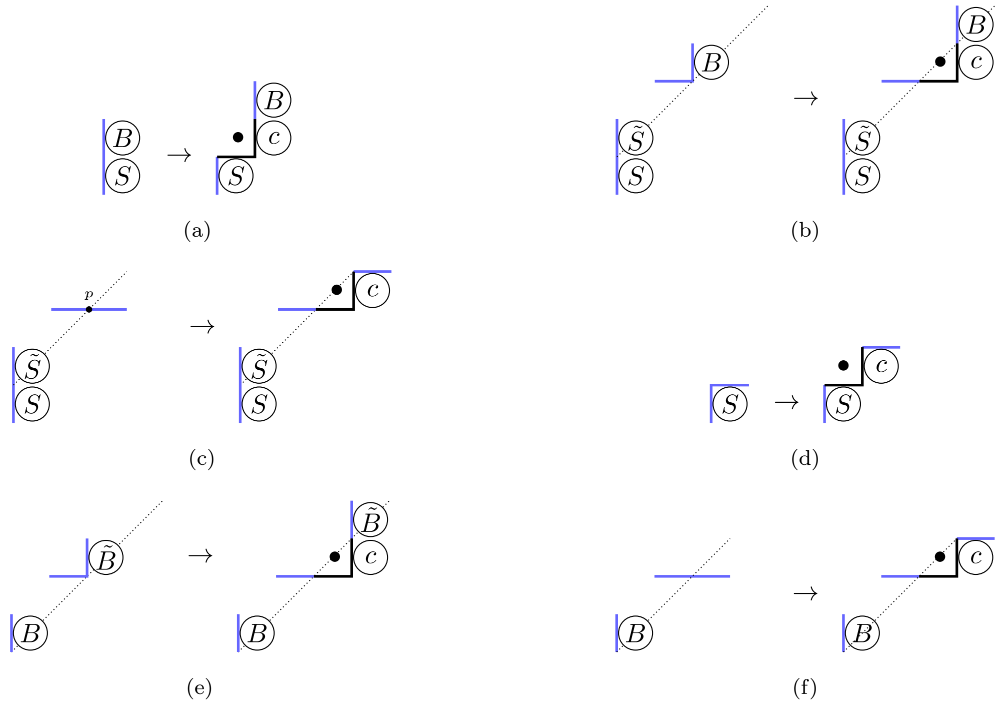


Fig. 5: Surjectivity for $i \geq s$.

First consider a dinv marker of the first kind, i.e. a label S at height $(i-s)$ smaller than c.

- If the dinv marker is followed by a vertical step whose label B is bigger than c, then insert the decorated label c directly north-east of S, right under B. See Figure 5a.
- Suppose that the dinv marker is followed by a vertical step whose label $\tilde{S}$ is smaller than c, and before the path crosses the $(i-s+1)$-th diagonal horizontally[1]there is dinv marker of the second kind, i.e. a label B bigger than c at height $i-s+1$. Insert the decorated c such that it lies right below this B. See Figure 5b.
- Suppose that the dinv marker is followed by a vertical step whose label $\tilde{S}$ is smaller than c, and there is no dinv marker of the second kind between $\tilde{S}$ and the point p where the path crosses the $(i-s+1)$-th diagonal horizontally. At p, insert a horizontal step followed by a decorated vertical step labelled c. See Figure 5c.
- Suppose that the dinv marker is followed by a horizontal step. Then insert the decorated label c in the square north-east of S. See Figure 5d.

Next, consider a dinv marker of the second kind, i.e. a label B at height $(i-s+1)$, bigger than c. Since $i \geq s$, we know the path will cross the $(i-s+1)$-th diagonal horizontally after the dinv marker.

- Suppose that before the path crosses the $(i-s+1)$-th diagonal horizontally there is a second dinv marker of the second kind labelled $\tilde{B}$. Insert the decorated c such that it lies right below $\tilde{B}$. See Figure 5e.
- Suppose that there is no dinv marker of the second kind between B and the point p where the path crosses the $(i-s+1)$-th diagonal horizontally. At p, insert a horizontal step followed by a decorated vertical step labelled c. See Figure 5f.

This completes the list of possibilities and thus the the argument for the bijectivity of the correspondence between $(i,c)^\bullet$-insertions with $i \geq s$ and interlacings of the $z_i^\bullet(c)$ inserted c's and $w_{i,s}^\bullet(c)$ dinv markers, starting with a dinv marker and without two consecutive c's. Each time a dinv marker precedes an inserted

[1] If the step labelled S is followed by another vertical step, the path crosses the $(i-s+1)$-th diagonal vertically. Thus, since $i \geq s$, the path will cross the same diagonal horizontally after the vertical crossing.

c a unit of dinv is created. Consider each of the $z_i^\bullet(c)$ inserted c's as part of a block with the dinv marker that precedes it. There is a unit of dinv for each block ($z_i^\bullet(c)$ units), a unit of dinv for each pair of blocks since the dinv marker of the first block creates dinv with the c of the second block ($\binom{z_i^\bullet(c)}{2}$ units), and a unit of dinv each time one of the $w_{i,s}^\bullet(c) - z_i^\bullet(c)$ dinv markers that are not part of a block precedes an inserted c (q-counted by $\begin{bmatrix} w_{i,s}^\bullet(c) \\ z_i^\bullet(c) \end{bmatrix}_q$). Lastly, all inserted decorated valleys contribute -1 to the dinv count ($-z_i^\bullet(c)$ in total). So the total contribution to the dinv by this insertion is q-counted by

$$q^{z_i^\bullet(c)} q^{\binom{z_i^\bullet(c)}{2}} q^{-z_i^\bullet(c)} \begin{bmatrix} w_{i,s}^\bullet(c) \\ z_i^\bullet(c) \end{bmatrix}_q = q^{\binom{z_i^\bullet(c)}{2}} \begin{bmatrix} w_{i,s}^\bullet(c) \\ z_i^\bullet(c) \end{bmatrix}_q .$$

Now suppose $i < s$. Using the same techniques as for the previous case, we will show that $(i,c)^\bullet$-insertions correspond bijectively to an interlacing of $w_{i,s}^\bullet(c)$ dinv markers and the $z_i^\bullet(c)$ inserted c's, *ending* with a dinv marker, and without two consecutive c's. For $i < s-1$ or $c \neq 0$ these are all the dinv markers, for $i = s-1$ and $c = 0$, $w_{i,s}^\bullet(c)$ equals the number of dinv markers minus 1.

Well defined. There are three things to show. First, that the interlacing corresponding to an insertion has no two consecutive c's. Exactly the same argument as for $i \geq s$ applies. Second, we show that the interlacing corresponding to any insertion ends with a dinv marker. Consider c an inserted label at height $i-s$. If the step c labels is followed by a vertical step, this must be labelled with a label bigger that c and so this is a dinv marker. Suppose that the inserted c is followed by a horizontal step.

Since the path must end east, there must be two consecutive vertical steps, at height $i-s$ and $i-s+1$, after c. If the label of the second of these steps is bigger than c it is a dinv marker. If not, the label S_1 of the first vertical step must be smaller than c, so if it is not decorated, it is a dinv marker. If it is decorated it may be preceded by a string of decorated valleys at height $i-s$, labelled $S_2, \dots, S_l$ with $S_1 > \cdots > S_l$ (by contractibility). The step labelled S_l is preceded by a horizontal step; if this step is preceded by a non-decorated vertical step its label must be smaller than c and is thus a dinv marker. If it is preceded by a second horizontal step we may deduce the existence of two consecutive vertical steps (at height $i-s$ and $i-s+1$) between c and S_l. We have arrived at the same situation as at the beginning of the paragraph. Since the path is finite this loop must terminate and a dinv marker exists after c.

Finally, for $i = s-1$ and $c = 0$, $w_{s-1,s}^\bullet(0)$ is equal to the number of dinv markers minus 1. We have that the interlacing between inserted c's and dinv marker must start with a dinv marker. Indeed, by definition the path may not start with a decorated 0 at height -1 so the first decorated 0 at height -1 must be preceded by two horizontal step and thus a positive label at height 0. Therefore, disregarding this first dinv marker of the interlacing, an $(s-1,0)^\bullet$-insertion corresponds to an interlacing of the $z_{s-1}^\bullet(0)$ inserted 0's and $w_{s-1,s}^\bullet(0)$ remaining dinv markers.

Remark 4 Keep in mind that this disregarded dinv marker creates $z_{s-1}^\bullet(0)$ units of dinv with all the 0's that follow it in the interlacing.

Injectivity. The argument is the same as for $i \geq s$.

Surjectivity. The fact that there must be a dinv marker to the right of all inserted c's ensures that the insertion algorithms for $i \geq s$ also apply here. So the only thing left to show is that, if $i \neq s-1$ or $c \neq 0$, we may always insert a decorated c to the left of all dinv markers. We consider the first label at height $i-s$, denote it F and consider the following cases.

- Suppose that F is a dinv marker or appears before all dinv markers, and is preceded by a horizontal step. Then this step must be preceded by another horizontal step, else the step labelled F would not be the first at its height. Insert a horizontal step followed by a decorated vertical step labelled c between these two horizontal steps. If F is decorated, the insertion order ensures that $c < F$ and so F labels a contractible valley. See Figure 6a.
- Suppose F is a dinv marker or appears before all dinv markers and is preceded by a vertical step. Since the path starts at $(0,0)$ this implies that before F there must be point were the path crosses the $(i-s)$-th diagonal horizontally. The two consecutive horizontal steps of this crossing must be preceded by a third horizontal step, since if there was a vertical step preceding them, F would not be the first label at its height. Insert a horizontal step followed by a decorated vertical step labelled c after the first (from the left) of these three horizontal steps. See Figure 6b.
- Suppose F is preceded by a dinv marker, a label $B > c$ at height $i-s+1$. Then the step labelled B must be preceded by a horizontal step, for if it were preceded by a vertical one, F would not be the first label at its height. Insert a horizontal step followed by a decorated vertical step labelled c after this horizontal step, underneath B. See Figure 6c.

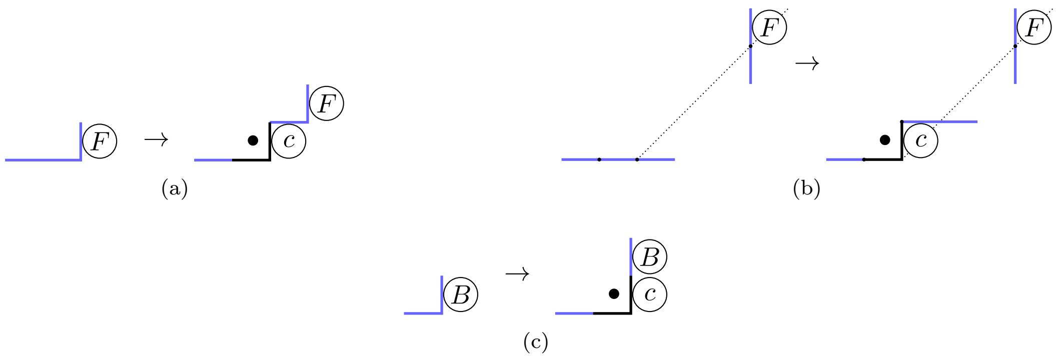


Fig. 6: Surjectivity for $i < s$.

So we proved the bijective correspondence between $(i,c)^\bullet$-insertions and interlacings of the $z_i^\bullet(c)$ inserted c's and $w_{i,s}^\bullet(c)$ dinv markers, ending with a dinv marker and without two consecutive c's. By definition each time a dinv marker precedes an inserted c, a unit of dinv is created.

Consider each of the $z_i^\bullet(c)$ inserted c's as part of a block with the dinv marker that *follows* it. As for the previous case there is a unit of dinv for each pair of blocks since the dinv marker of the first block creates dinv with the c of the second block ($\binom{z_i^\bullet(c)}{2}$ units), and a unit of dinv each time one of the $w_{i,s}^\bullet(c) - z_i^\bullet(c)$ dinv markers that are not part of a block precedes an inserted c (q-counted by $\begin{bmatrix} w_{i,s}^\bullet(c) \\ z_i^\bullet(c) \end{bmatrix}_q$). Also all inserted decorated valleys contribute -1 to the dinv count ($-z_i^\bullet(c)$ in total). Furthermore, for $c \neq 0$ and $i < s$ any $(i,c)^\bullet$-insertion creates $z_i^\bullet(c)$ units of bonus dinv. Lastly, we must not forget the $z_{s-1}^\bullet(0)$ units of primary dinv created with the first dinv marker and the 0's at height -1 (see Remark 4). It follows that the contribution to the dinv for all possible $(i,c)^\bullet$-insertions is q-counted by

$$q^{z_{s-1}^\bullet(0)\delta_{i,s-1}\delta_{c,0}} q^{(1-\delta_{c,0})z_i^\bullet(c)} q^{-z_i^\bullet(c)} q^{\binom{z_i^\bullet(c)}{2}} \begin{bmatrix} w_{i,s}^\bullet \\ z_i^\bullet(c) \end{bmatrix}_q$$

See Figures 11 and 12in Appendix A.

Taking the product over all possible i and c and using

$$\begin{aligned}
&\sum_{i<s}\sum_{c\in\mathbb{N}} \left((1-\delta_{c,0})z_i(c) + z_{s-1}^\bullet(0)\delta_{i,s-1}\delta_{c,0} + (1-\delta_{c,0})z_i^\bullet(c) - z_i^\bullet(c)\right) \\
&= \sum_{i<s}\sum_{c>0} \left(z_i(c) + z_i^\bullet(c) - z_i^\bullet(c)\right) + \sum_{i<s-1} (-z_i^\bullet(0)) + z_{s-1}^\bullet(0) - z_{s-1}^\bullet(0) \\
&= \sum_{c>0}\sum_{i<s} z_i(c) + \sum_{i<s-1} (-z_i^\bullet(0)) = b(z,s).
\end{aligned}$$

we finally obtain the announced formula.

6 The valley Delta implies the valley square

Let us define

$$\mathsf{LSQ}_{q,t;x}(z,s) \coloneqq \sum_{\substack{\pi \in \mathsf{LSQ}(m,n)^{\bullet k} \\ \mathsf{shift}(\pi)=s \\ \mathsf{dw}(\pi)=z}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi,$$

where z is the diagonal word of some decorated square path in $\mathsf{LSQ}(m,n)^{\bullet k}$. Let $r_i \coloneqq \sum_{c>0} z_i(c)$. We want to relate $\mathsf{LSQ}_{q,t;x}(z,s)$ and $\mathsf{LSQ}_{q,t;x}(z,s')$. Let us state some lemmas.

Lemma 1 *For z diagonal word of a path in $\mathsf{LSQ}'(m,n)^{\bullet k}$ with $\ell+1$ runs, and $0 < s \leq \ell$, we have*

$$\prod_{c\geq 0} \frac{\begin{bmatrix} w_{s,s}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s-1,s-1}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q} = \frac{[r_s]_q}{[r_{s-1}]_q} \cdot \frac{[r_s + z_s(0) - 1]_q!}{[r_{s-1} + z_{s-1}(0) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}.$$

Proof Recall that

$$w_{s,s}(c) = 1 - \delta_{c,0} + \sum_{a>c} z_s(a), \qquad w_{s-1,s-1}(c) = 1 - \delta_{c,0} + \sum_{a>c} z_{s-1}(a).$$

We have

$$\prod_{c\geq 0} \frac{\begin{bmatrix} w_{s,s}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s-1,s-1}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q} = \prod_{c\geq 0} \frac{[w_{s,s}(c)+z_s(c)-1]_q!}{[w_{s,s}(c)-1]_q!} \frac{[w_{s-1,s-1}(c)-1]_q!}{[w_{s-1,s-1}(c)+z_{s-1}(c)-1]_q!} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}$$

$$= \prod_{c\geq 0} \frac{[\sum_{a\geq c} z_s(a) - \delta_{c,0}]_q!}{[\sum_{a>c} z_s(a) - \delta_{c,0}]_q!} \frac{[\sum_{a>c} z_{s-1}(a) - \delta_{c,0}]_q!}{[\sum_{a\geq c} z_{s-1}(a) - \delta_{c,0}]_q!} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}$$

$$= \frac{[\sum_{a\geq 0} z_s(a) - 1]_q!}{[\sum_{a>0} z_s(a) - 1]_q!} \frac{[\sum_{a>0} z_{s-1}(a) - 1]_q!}{[\sum_{a\geq 0} z_{s-1}(a) - 1]_q!} \cdot \prod_{c>0} \frac{[\sum_{a\geq c} z_s(a)]_q!}{[\sum_{a>c} z_s(a)]_q!} \frac{[\sum_{a>c} z_{s-1}(a)]_q!}{[\sum_{a\geq c} z_{s-1}(a)]_q!} \cdot \prod_{c\geq 0} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}$$

$$= \frac{[r_s + z_s(0) - 1]_q! [r_{s-1} - 1]_q!}{[r_s - 1]_q! [r_{s-1} + z_{s-1}(0) - 1]_q!} \frac{[\sum_{a>0} z_s(a)]_q!}{[\sum_{a>0} z_{s-1}(a)]_q!} \cdot \prod_{c>0} \frac{[\sum_{a>c} z_s(a)]_q!}{[\sum_{a>c} z_s(a)]_q!} \frac{[\sum_{a>c} z_{s-1}(a)]_q!}{[\sum_{a>c} z_{s-1}(a)]_q!} \cdot \prod_{c\geq 0} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}$$

$$= \frac{[r_s + z_s(0) - 1]_q! [r_{s-1} - 1]_q!}{[r_s - 1]_q! [r_{s-1} + z_{s-1}(0) - 1]_q!} \cdot \frac{[r_s]_q!}{[r_{s-1}]_q!} \cdot \prod_{c\geq 0} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}$$

$$= \frac{[r_s]_q}{[r_{s-1}]_q} \cdot \frac{[r_s + z_s(0) - 1]_q!}{[r_{s-1} + z_{s-1}(0) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}$$

and it is not hard to check that all the denominators are non-zero as they are either q-analogues of positive integers or q-factorials of non-negative integers. This completes the proof.

Lemma 2 *For z diagonal word of a path in $\mathsf{LSQ}'(m,n)^{\bullet k}$ with $\ell + 1$ runs, and $0 < s \leq \ell$, we have*

$$\prod_{c\geq 0} \frac{\begin{bmatrix} w_{s-1,s}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s,s-1}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q} = \frac{[r_{s-1} + z_{s-1}(0) - 1]_q!}{[r_s + z_s(0) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!}.$$

Proof Recall that

$$w_{s-1,s}(c) = w_{s,s-1}(c) = \sum_{a>c} z_s(a) + \sum_{a<c} z_{s-1}(a).$$

Let $m = \max\{c \geq 0 \mid z_s(c) > 0 \text{ or } z_{s-1}(c) > 0\}$. We have

$$\begin{aligned}
\prod_{c\geq 0} \frac{\begin{bmatrix} w_{s-1,s}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s,s-1}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q} &= \prod_{c\geq 0} \frac{[w_{s-1,s}(c)+z_{s-1}(c)-1]_q!}{[w_{s-1,s}(c)-1]_q!} \frac{[w_{s,s-1}(c)-1]_q!}{[w_{s,s-1}(c)+z_s(c)-1]_q!} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \prod_{c\geq 0} \frac{[w_{s-1,s}(c)+z_{s-1}(c)-1]_q!}{[w_{s,s-1}(c)+z_s(c)-1]_q!} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \prod_{c\geq 0} \frac{[\sum_{a>c} z_s(a) + \sum_{a\leq c} z_{s-1}(a) - 1]_q!}{[\sum_{a\geq c} z_s(a) + \sum_{a<c} z_{s-1}(a) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{\prod_{c=0}^{m} [\sum_{a>c} z_s(a) + \sum_{a\leq c} z_{s-1}(a) - 1]_q!}{\prod_{c=0}^{m} [\sum_{a>c-1} z_s(a) + \sum_{a\leq c-1} z_{s-1}(a) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{\prod_{c=0}^{m} [\sum_{a>c} z_s(a) + \sum_{a\leq c} z_{s-1}(a) - 1]_q!}{\prod_{c=-1}^{m-1} [\sum_{a>c} z_s(a) + \sum_{a\leq c} z_{s-1}(a) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{[\sum_{a>m} z_s(a) + \sum_{a\leq m} z_{s-1}(a) - 1]_q!}{[\sum_{a>-1} z_s(a) + \sum_{a\leq -1} z_{s-1}(a) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{[\sum_{a\leq m} z_{s-1}(a) - 1]_q!}{[\sum_{a\geq 0} z_s(a) - 1]_q!} \cdot \prod_{c\geq 0} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{[r_{s-1} + z_{s-1}(0) - 1]_q!}{[r_s + z_s(0) - 1]_q!} \cdot \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!}
\end{aligned}$$

and once again it is not hard to check that all the denominators are non-zero as they are q-factorials of non-negative integers. This completes the proof.

Lemma 3 *For z diagonal word of a path in $\mathsf{LSQ}'(m,n)^{\bullet k}$ with $\ell+1$ runs, and $0 \leq s' < s \leq \ell$, we have*

$$\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w_{i,s}(c)+z_i(c)-1 \\ z_i(c)\end{bmatrix}_q = \frac{[r_s]_q}{[r_{s'}]_q}\cdot \prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w_{i,s'}(c)+z_i(c)-1 \\ z_i(c)\end{bmatrix}_q.$$

Proof Let us preliminarily prove the result for $s'=s-1$. By definition we have that $w_{i,s}(c)=w_{i,s-1}(c)$ for $i\notin\{s-1,s\}$, so it is enough to show that

$$\begin{aligned}\prod_{c\geq 0}&\begin{bmatrix} w_{s-1,s}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c)\end{bmatrix}_q \begin{bmatrix} w_{s,s}(c)+z_s(c)-1 \\ z_s(c)\end{bmatrix}_q \\ &= \frac{[r_s]_q}{[r_{s-1}]_q}\cdot\prod_{c\geq 0}\begin{bmatrix} w_{s-1,s-1}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c)\end{bmatrix}_q \begin{bmatrix} w_{s,s-1}(c)+z_s(c)-1 \\ z_s(c)\end{bmatrix}_q.\end{aligned} \tag{1}$$

By Lemma 1 we have

$$\prod_{c\geq 0}\begin{bmatrix} w_{s,s}(c)+z_s(c)-1 \\ z_s(c)\end{bmatrix}_q = \frac{[r_s]_q}{[r_{s-1}]_q}\cdot\frac{[r_s+z_s(0)-1]_q!}{[r_{s-1}+z_{s-1}(0)-1]_q!}\cdot\prod_{c\geq 0}\frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}\cdot\begin{bmatrix} w_{s-1,s-1}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c)\end{bmatrix}_q$$

and by Lemma 2 we have

$$\prod_{c\geq 0}\begin{bmatrix} w_{s-1,s}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c)\end{bmatrix}_q = \frac{[r_{s-1}+z_{s-1}(0)-1]_q!}{[r_s+z_s(0)-1]_q!}\cdot\prod_{c\geq 0}\frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!}\cdot\begin{bmatrix} w_{s,s-1}(c)+z_s(c)-1 \\ z_s(c)\end{bmatrix}_q$$

so, taking the product of the two expressions, after the obvious simplifications the statement for $s'=s-1$ follows.

Now, applying Equation 1 repeatedly, we get

$$\begin{aligned}\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w_{i,s}(c)+z_i(c)-1 \\ z_i(c)\end{bmatrix}_q &= \frac{[r_s]_q}{[r_{s-1}]_q}\cdot\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w_{i,s-1}(c)+z_i(c)-1 \\ z_i(c)\end{bmatrix}_q \\ &= \frac{[r_s]_q}{[r_{s-1}]_q}\frac{[r_{s-1}]_q}{[r_{s-2}]_q}\cdot\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w_{i,s-2}(c)+z_i(c)-1 \\ z_i(c)\end{bmatrix}_q \\ &= \dots \\ &= \frac{[r_s]_q}{[r_{s-1}]_q}\cdots\frac{[r_{s'+1}]_q}{[r_{s'}]_q}\cdot\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w_{i,s'}(c)+z_i(c)-1 \\ z_i(c)\end{bmatrix}_q \\ &= \frac{[r_s]_q}{[r_{s'}]_q}\cdot\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w_{i,s'}(c)+z_i(c)-1 \\ z_i(c)\end{bmatrix}_q\end{aligned}$$

as desired.

Lemma 4 *For z diagonal word of a path in $\mathsf{LSQ}'(m,n)^{\bullet k}$ with $\ell+1$ runs, and $0\leq s'<s\leq\ell$, if $r_s - z^{\bullet}_{s-1}(0)>0$ and $r_{s'}-z^{\bullet}_{s'-1}(0)>0$, we have*

$$\frac{[r_s]_q}{[r_s - z^{\bullet}_{s-1}(0)]_q}\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w^{\bullet}_{i,s}(c) \\ z^{\bullet}_i(c)\end{bmatrix}_q = \frac{[r_{s'}]_q}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q}\prod_{i=0}^{\ell}\prod_{c\geq 0}\begin{bmatrix} w^{\bullet}_{i,s'}(c) \\ z^{\bullet}_i(c)\end{bmatrix}_q.$$

Proof Let us recall that

$$w^{\bullet}_{i,s}(c) \coloneqq \sum_{a<c} z_i(a) + \sum_{a>c} z_{i+1}(a) - \delta_{c,0}\delta_{i,s-1}.$$

For $c \neq 0$, $w^\bullet_{i,s}(c)$ does not depend on s, and $w^\bullet_{i,s}(0) = w^\bullet_{i,s'}(0) \geq 0$ for $i \notin \{s-1, s'-1\}$, so it is enough to show that

$$\frac{[r_s]_q}{[r_s - z^\bullet_{s-1}(0)]_q} \begin{bmatrix} w^\bullet_{s-1,s}(0) \\ z^\bullet_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^\bullet_{s'-1,s}(0) \\ z^\bullet_{s'-1}(0) \end{bmatrix}_q = \frac{[r_{s'}]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \begin{bmatrix} w^\bullet_{s-1,s'}(0) \\ z^\bullet_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^\bullet_{s'-1,s'}(0) \\ z^\bullet_{s'-1}(0) \end{bmatrix}_q.$$

We have

$$w^\bullet_{s-1,s}(0) = r_s - 1 \qquad w^\bullet_{s'-1,s}(0) = r_{s'} \qquad w^\bullet_{s-1,s'}(0) = r_s \qquad w^\bullet_{s'-1,s'}(0) = r_{s'} - 1$$

so

$$\begin{aligned}
\frac{[r_s]_q}{[r_s - z^\bullet_{s-1}(0)]_q} \begin{bmatrix} w^\bullet_{s-1,s}(0) \\ z^\bullet_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^\bullet_{s'-1,s}(0) \\ z^\bullet_{s'-1}(0) \end{bmatrix}_q &= \frac{[r_s]_q}{[r_s - z^\bullet_{s-1}(0)]_q} \begin{bmatrix} r_s - 1 \\ z^\bullet_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} r_{s'} \\ z^\bullet_{s'-1}(0) \end{bmatrix}_q \\
&= \frac{[r_s]_q}{[r_s - z^\bullet_{s-1}(0)]_q} \frac{[r_s - 1]_q!}{[r_s - z^\bullet_{s-1}(0) - 1]_q! [z^\bullet_{s-1}(0)]_q!} \frac{[r_{s'}]_q!}{[r_{s'} - z^\bullet_{s'-1}(0)]_q! [z^\bullet_{s'-1}(0)]_q!} \\
&= \frac{[r_s]_q!}{[r_s - z^\bullet_{s-1}(0)]_q! [z^\bullet_{s-1}(0)]_q!} \frac{[r_{s'}]_q!}{[r_{s'} - z^\bullet_{s'-1}(0)]_q! [z^\bullet_{s'-1}(0)]_q!} \\
&= \frac{[r_{s'}]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \frac{[r_s]_q!}{[r_s - z^\bullet_{s-1}(0)]_q! [z^\bullet_{s-1}(0)]_q!} \frac{[r_{s'} - 1]_q!}{[r_{s'} - z^\bullet_{s'-1}(0) - 1]_q! [z^\bullet_{s'-1}(0)]_q!} \\
&= \frac{[r_{s'}]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \begin{bmatrix} r_s \\ z^\bullet_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} r_{s'} - 1 \\ z^\bullet_{s'-1}(0) \end{bmatrix}_q \\
&= \frac{[r_{s'}]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \begin{bmatrix} w^\bullet_{s-1,s'}(0) \\ z^\bullet_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^\bullet_{s'-1,s'}(0) \\ z^\bullet_{s'-1}(0) \end{bmatrix}_q
\end{aligned}$$

as desired.

Theorem 6 *Let z be the diagonal word of a path in $\mathsf{LSQ}'(m,n)^{\bullet k}$ with $\ell+1$ runs, and $0 \leq s' < s \leq \ell$. We have*

$$[r_{s'} - z^\bullet_{s'-1}(0)]_q \cdot \mathsf{LSQ}_{q,t;x}(z,s) = q^{\sum_{i=s'}^{s-1}(r_i - z^\bullet_{i-1}(0))} [r_s - z^\bullet_{s-1}(0)]_q \cdot \mathsf{LSQ}_{q,t;x}(z,s').$$

Proof First of all notice that $r_i - z^\bullet_{i-1}(0) \geq 0$, and if the path has shift s, then $r_s - z^\bullet_{s-1}(0) > 0$. In fact, between any two decorated zero valleys in the $(i-s-1)$-th diagonal there must be a rise (that contributes to r_i). If $i \leq s$ there must be a rise after the last such valley. If $i \geq s$ there must be a non-decorated positive label in the $(i-s)$-th diagonal before the first such valley (as it must be preceded by two horizontal steps). For $i = s$ both hold, and this gives the strict inequality. Because of this, either $r_s - z^\bullet_{s-1}(0) > 0$ or $\mathsf{LSQ}(z,s) = \varnothing$, in which case the equality $0 = 0$ trivially holds. The same argument applies to s', so from now on, we assume $r_s - z^\bullet_{s-1}(0) \neq 0$ and $r_{s'} - z^\bullet_{s'-1}(0) \neq 0$.

Now, via a simple calculation we get $b(z,s) - b(z,s') = \sum_{i=s'}^{s-1}(r_i - z^\bullet_{i-1}(0))$. Moreover, by Lemma 3 we have

$$\prod_{i=0}^{\ell} \prod_{c \geq 0} \begin{bmatrix} w_{i,s}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q = \frac{[r_s]_q}{[r_{s'}]_q} \cdot \prod_{i=0}^{\ell} \prod_{c \geq 0} \begin{bmatrix} w_{i,s'}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q$$

and by Lemma 4 we have

$$\frac{[r_s]_q}{[r_s - z^\bullet_{s-1}(0)]_q} \prod_{i=0}^{\ell} \prod_{c \geq 0} \begin{bmatrix} w^\bullet_{i,s}(c) \\ z^\bullet_i(c) \end{bmatrix}_q = \frac{[r_{s'}]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \prod_{i=0}^{\ell} \prod_{c \geq 0} \begin{bmatrix} w^\bullet_{i,s'}(c) \\ z^\bullet_i(c) \end{bmatrix}_q.$$

By combining the two with Theorem 5, we get

$$\mathsf{LSQ}_{q,t;x}(z,s) = q^{\sum_{i=s'}^{s-1}(r_i - z^\bullet_{i-1}(0))} \frac{[r_s - z^\bullet_{s-1}(0)]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \cdot \mathsf{LSQ}_{q,t;x}(z,s').$$

which is exactly what we wanted to show.

Corollary 2 *If $r_0 \neq 0$, then*

$$\mathsf{LSQ}_{q,t;x}(z,s) = q^{b(z,s)} \frac{[r_s - z^{\bullet}_{s-1}(0)]_q}{[r_0]_q} \mathsf{LD}_{q,t;x}(z),$$

where we used the obvious notation $\mathsf{LD}_{q,t;x}(z) = \mathsf{LSQ}_{q,t;x}(z,0)$.

Proof It follows immediately by applying Theorem 6 with $s' = 0$ (notice that $z^{\bullet}_{-1}(0) = 0$).

Corollary 3

$$\mathsf{LSQ}'_{q,t;x}(m, n\backslash r)^{\bullet k} = \frac{[n-k]_q}{[r]_q} \mathsf{LD}_{q,t;x}(m, n\backslash r)^{\bullet k}$$

Proof Given z diagonal word of a path in $\mathsf{LSQ}'(m,n)^{\bullet k}$ with $\ell+1$ runs, we have

$$\begin{aligned}
\sum_{s=0}^{\ell} \mathsf{LSQ}_{q,t;x}(z,s) &= \sum_{s=0}^{\ell} q^{b(z,s)} \frac{[r_s - z^{\bullet}_{s-1}(0)]_q}{[r_0]_q} \mathsf{LD}_{q,t;x}(z) \\
&= \frac{\sum_{s=0}^{\ell} q^{b(z,s)} [r_s - z^{\bullet}_{s-1}(0)]_q}{r_0]_q} \mathsf{LD}_{q,t;x}(z) \\
&= \frac{[\sum_{s=0}^{\ell} (r_s - z^{\bullet}_{s-1}(0))]_q}{[r_0]_q} \mathsf{LD}_{q,t;x}(z)
\end{aligned}$$

and now taking the sum over all such z with $r_0 = r$, since $\sum_{s=0}^{\ell} r_s = n - k + \sum_{s=0}^{\ell} z^{\bullet}_{s-1}(0)$ (the total number of non-decorated positive labels), the thesis follows immediately.

Theorem 7 (Conditional modified Delta square conjecture, valley version) *If Conjecture 7 holds, then so does Conjecture 9. As a special case, if Conjecture 6 holds, then so does Conjecture 8.*

Proof We recall the statement of Conjecture 7, which is

$$\Delta_{h_m} \Theta_{e_k} \nabla E_{n-k,r} = \sum_{\pi \in \mathsf{LD}(m,n\backslash r)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi}.$$

Applying Corollary 3, we have

$$\frac{[n-k]_q}{[r]_q} \Delta_{h_m} \Theta_{e_k} \nabla E_{n-k,r} = \sum_{\pi \in \mathsf{LSQ}'(m,n\backslash r)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi}.$$

Taking the sum over r and using Proposition 2, we get

$$\Delta_{h_m} \Theta_{e_k} \nabla \omega(p_{n-k}) = \sum_{\pi \in \mathsf{LSQ}'(m,n)^{\bullet k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi},$$

as desired.

By Remark 2, the case $k = 0$ of Conjecture 7 holds. So by Theorem 7, the case $k = 0$ of Conjecture 9 holds as well.

Corollary 4 (Generalised square theorem)

$$\Delta_{h_m} \nabla \omega(p_n) = \sum_{\pi \in \mathsf{LSQ}(m,n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi}.$$

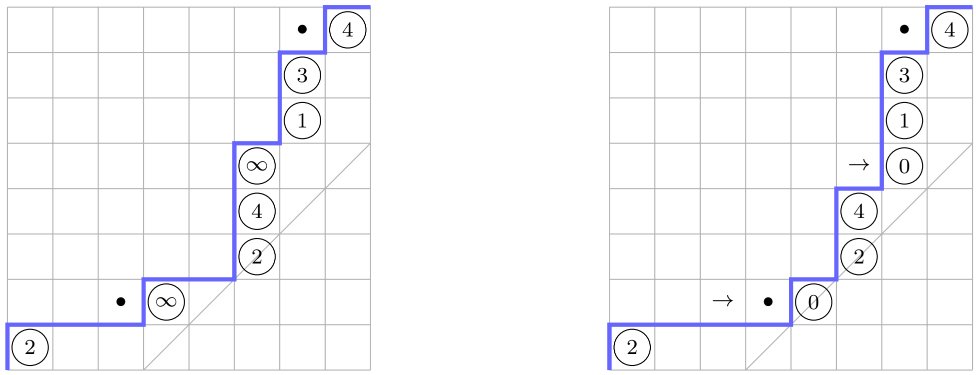


Fig. 7: “Pushing” of ∞’s.

7 Concluding remarks

As we mentioned before, the slightly contrived conditions on the positions of the steps labelled with zeros can be reformulated quite naturally by considering a step labelled 0 as the “pushing” of a step labelled ∞.

Performing this manoeuvre does not change the dinv (if we define that the ∞’s under the main diagonal do not contribute to the bonus dinv and that there are no ∞’s on the base diagonal). The area changes by a constant factor equal to the number of zeros.

Several open problems arise from our discussion. There is no interpretation of the symmetric function $\Delta_{h_m} \Theta_{e_k} \nabla \omega(p_{n-k})$ in terms of rise-decorated square paths, for which also the schedule formula is lacking. This is one of the very few instances where the valley version seems to be easier to treat than the rise version. Understanding the rise version better might lead to a unified valley-rise conjecture interpreting $\Theta_{e_j} \Theta_{e_k} \nabla e_{n-k-j}$.

Lastly, it would be nice to show that the valley Delta conjecture implies the generalised valley Delta conjecture. Given that, our results would be conditional only on the valley Delta conjecture. There might be a way to prove this using the “pushing” manoeuvre described above to interpret the behaviour of the $h_j^\perp$ operator. We have some symmetric function identities suggesting that this avenue might be fruitful, and some of these conjectural identities are strongly suggested by certain relations among the combinatorial objects.

A Figures for schedule numbers

This appendix contains figures illustrating the construction of some of the square paths of $\mathsf{LSQ}(1,8)^{\bullet 2}$ with diagonal word $44\ 223\ \dot{0}11\dot{2}$ and shift 1. They serve as visuals for the proof of Theorem 5.

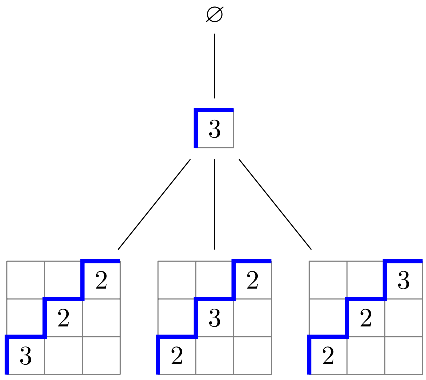


Fig. 8: $(1,3)$ and $(1,2)$-insertion

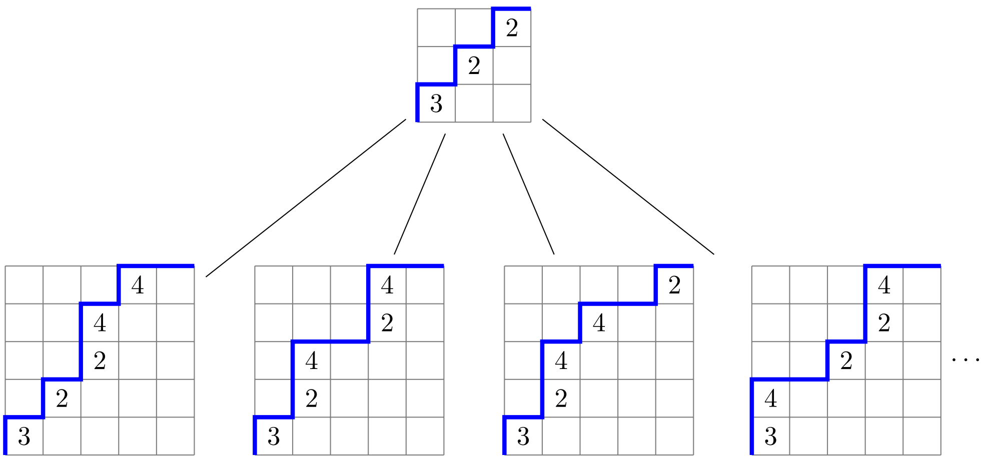


Fig. 9: $(2,4)$-insertion

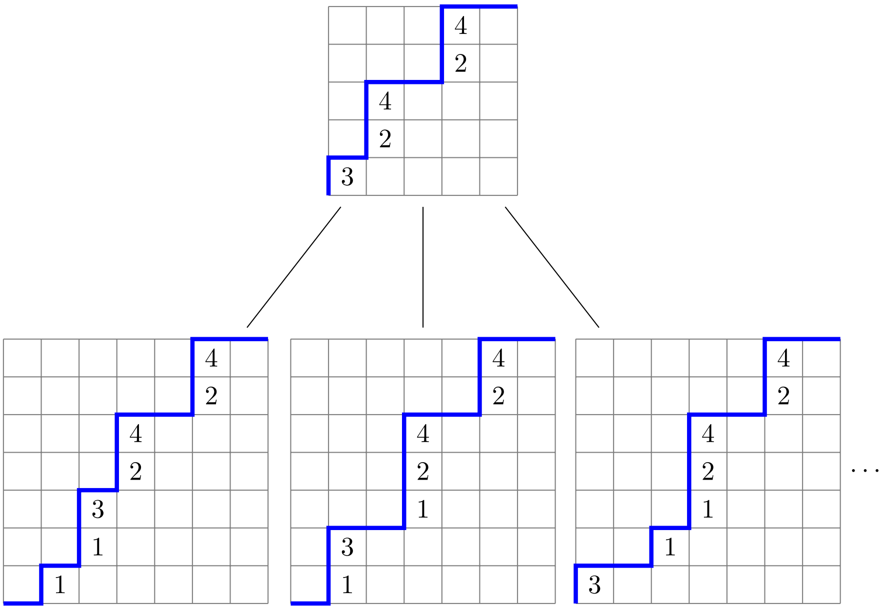


Fig. 10: $(0,1)$-insertion

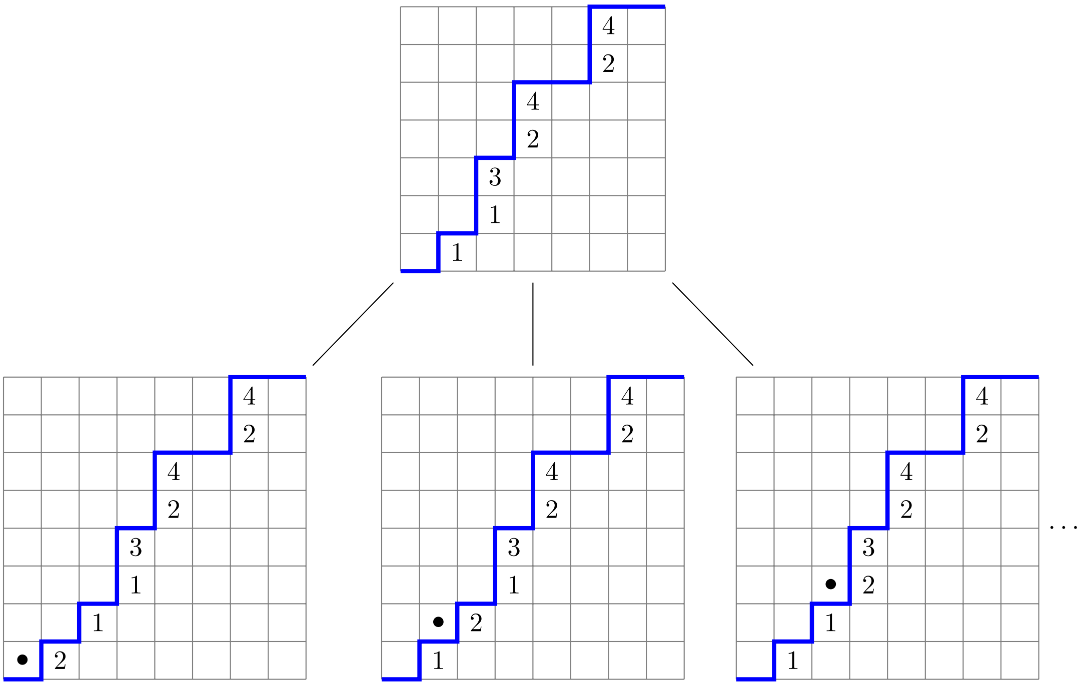


Fig. 11: $(0,2)^{\bullet}$-insertion

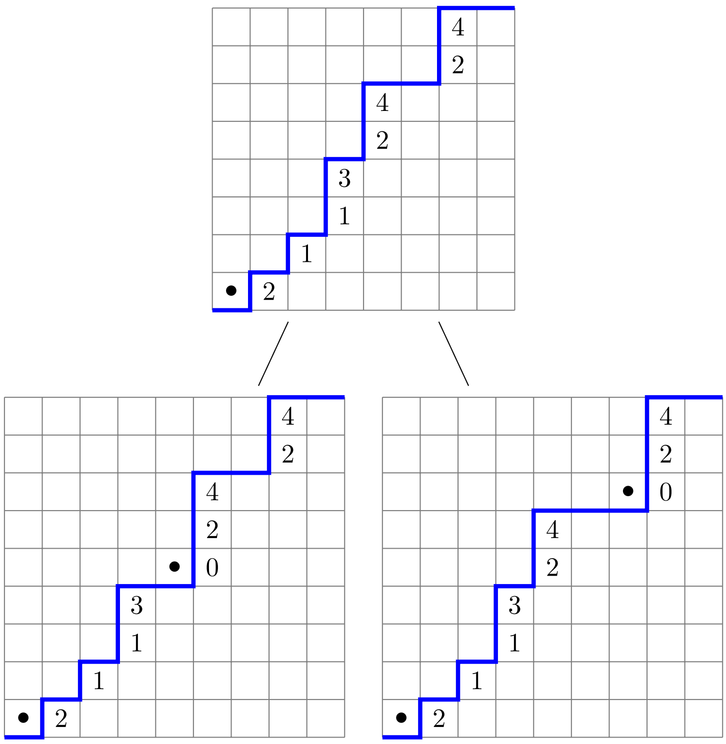


Fig. 12: $(0,0)^{\bullet}$-insertion

References

[1] François Bergeron and Adriano M. Garsia, *Science fiction and Macdonald's polynomials*, Algebraic methods and q-special functions (montréal, QC, 1996), 1999, pp. 1–52. MR1726826

[2] François Bergeron, Adriano M. Garsia, Mark Haiman, and Glenn Tesler, *Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions*, Methods Appl. Anal. **6** (1999), no. 3, 363–420. Dedicated to Richard A. Askey on the occasion of his 65th birthday, Part III. MR1803316

[3] Mahir Can and Nicholas Loehr, *A proof of the q,t-square conjecture*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1419–1434. MR2259069

[4] Erik Carlsson and Anton Mellit, *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31** (2018), no. 3, 661–697. MR3787405

[5] Michele D'Adderio, Alessandro Iraci, and Anna Vanden Wyngaerd, *Decorated Dyck paths, polyominoes, and the Delta conjecture*, Mem. Amer. Math. Soc. (2018).

[6] ———, *The Delta Square Conjecture*, International Mathematics Research Notices (201903), available at `rnz057.pdf`.

[7] ———, *The generalized Delta conjecture at t=0*, arXiv e-prints (2019Jan), arXiv:1901.02788, available at `1901.02788`.

[8] ———, *The Schröder case of the generalized Delta conjecture*, European Journal of Combinatorics **81** (2019), 58 –83.

[9] ———, *Theta operators, refined Delta conjectures, and coinvariants*, arXiv e-prints (2019Jun), available at `1906.02623`.

[10] Adriano Garsia, Jim Haglund, Jeffrey B. Remmel, and Meesue Yoo, *A Proof of the Delta Conjecture When $q = 0$*, Annals of Combinatorics **23** (2019Jun), no. 2, 317–333.

[11] James Haglund, *The q,t-Catalan numbers and the space of diagonal harmonics*, University Lecture Series, vol. 41, American Mathematical Society, Providence, RI, 2008. With an appendix on the combinatorics of Macdonald polynomials. MR2371044

[12] James Haglund, Mark Haiman, Nicholas Loehr, Jeffrey B. Remmel, and Anatoly Ulyanov, *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126** (2005), no. 2, 195–232. MR2115257

[13] James Haglund, Jeffrey B. Remmel, and Andrew T. Wilson, *The Delta Conjecture*, Trans. Amer. Math. Soc. **370** (2018), no. 6, 4029–4057. MR3811519

[14] James Haglund, Brendon Rhoades, and Mark Shimozono, *Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture*, Adv. Math. **329** (2018), 851–915. MR3783430

[15] James Haglund and Emily Sergel, *Schedules and the Delta Conjecture*, arXiv e-prints (2019Aug), available at `1908.04732`.

[16] Mark Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), no. 4, 941–1006. MR1839919

[17] ———, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), no. 2, 371–407. MR1918676

[18] Nicholas A. Loehr and Gregory S. Warrington, *Square q,t-lattice paths and $\nabla(p_n)$*, Trans. Amer. Math. Soc. **359** (2007), no. 2, 649–669. MR2255191

[19] Ian G. Macdonald, *Symmetric functions and Hall polynomials*, Second, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky, Oxford Science Publications. MR1354144

[20] Dun Qiu and Andrew Timothy Wilson, *The valley version of the Extended Delta Conjecture*, 2019.

[21] Jeffrey B. Remmel and Andrew T. Wilson, *An extension of MacMahon's equidistribution theorem to ordered set partitions*, J. Combin. Theory Ser. A **134** (2015), 242–277. MR3345306

[22] Brendon Rhoades, *Ordered set partition statistics and the Delta Conjecture*, J. Combin. Theory Ser. A **154** (2018), 172–217. MR3718065

[23] Emily Sergel, *A proof of the Square Paths Conjecture*, J. Combin. Theory Ser. A **152** (2017), 363–379. MR3682738

[24] Andrew T. Wilson, *An extension of MacMahon's equidistribution theorem to ordered multiset partitions*, Electron. J. Combin. **23** (2016), no. 1, Paper 1.5, 21. MR3484710

[25] Mike Zabrocki, *A proof of the 4-variable Catalan polynomial of the Delta conjecture*, ArXiv e-prints (September 2016), available at `1609.03497`.

[26] Mike Zabrocki, *A module for the Delta conjecture*, arXiv e-prints (2019Feb), arXiv:1902.08966, available at `1902.08966`.